Journal of Global Optimization **15:** 109–126, 1999.

A Geometric Approach to Global Optimization

STEFAN NICKEL and ANITA SCHÖBEL
Fachbereich Mathematik, Universität Kaiserslautern, D-67653 Kaiserslautern, Germany (e-mail: nickel@mathematik.uni-kl.de)

(Received 10 September 1997; accepted in revised form 21 October 1998)

Abstract. In this paper we consider the problem of optimizing a piecewise-linear objective function over a non-convex domain. In particular we do not allow the solution to lie in the interior of a prespecified region R. We discuss the geometrical properties of this problems and present algorithms based on combinatorial arguments. In addition we show how we can construct quite complicated shaped sets R while maintaining the combinatorial properties.

Key words: Reverse convex constraints, Geometric approach, Discretization, Piecewise linear programs

1. Introduction

The solution of piecewise linear programs has always played a special role in optimization. In general, piecewise-linearities are often employed to give a more realistic description of costs than can be achieved by linear terms alone. Also in worst case analysis piecewise linearity arises naturally. Furthermore, piecewise-linear programs may be used for approximating convex objective functions by piecewise linear functions. Such an approximation method including error bound analysis was proposed by Burkard et al. [1]. In addition, piecewise linear programs arise in specific branches of operations research, like for example location theory, where – given a number of existing facilities – a new facility has to be located so that some objectives are optimized. The objectives are often in the form of piecewise linear functions due to the fact that typically the sum (or the maximum) of weighted distances from the existing facilities to the new facility is chosen as a criterion. Moreover, these distances are often derived from norms with a polyhedral unit ball. See [4, 14 and 16] for various piecewise linear models in location theory. For an overview about methods for solving piecewise linear programs the reader is referred to [5–7], and references therein. As a conclusion it is quite natural also to look at global optimization problems with piecewise-linearities. In particular, we are interested in the problem of optimizing a piecewise-linear objective function over a non-convex domain. These includes problems with reverse convex constraints. For more details and an introduction to global optimization the reader is referred to [10, 11] and references therein. Now we will formally introduce the model we consider in this paper.

1.1. THE MODEL

Consider the following piecewise linear optimization problem in $\mathbb{R}^n$:

$$\text{(OL)} \ \min_{x \in \mathbb{R}^n} f(x) \ ,$$

where $f(x) := \max\{f_1(x), f_2(x), \ldots, f_K(x)\}$ and the $f_i, i = 1, 2, \ldots, K$ are K different affine linear functions. This is a well-known convex problem and can efficiently be solved by linear programming methods.

We can also easily introduce a feasibility region A, where A is a polyhedral set with

$$A := \{x \in \mathbb{R}^n : h_j(x) \leqslant p_j, \ j = 1, 2, \ldots, L\}.$$

Also (OL) with the additional restriction that $x \in A$ can be formulated as a linear program and therefore be solved efficiently.

$$\begin{aligned} &\min z \\ s.t. \quad &f_i(x) \leqslant z \quad i = 1, 2, \ldots, K \\ &h_j(x) \leqslant p_j \quad j = 1, 2, \ldots, L \end{aligned}$$

If, however, the feasible region A cannot be described by linear inequalities, or is not convex, we are in the area of global optimization. In particular we consider in this paper $A := \mathrm{cl}(\mathbb{R}^n \setminus R)$, with $R \subseteq \mathbb{R}^n$. By $\mathrm{cl}(S)$ we denote the closure of a set S. Now the minimization problem (ROL) reads

$$\min_{x \in A} f(x) \quad \text{or} \quad \min_{x \notin int(R)} f(x) \ ,$$

where $\mathrm{int}(R)$ denotes the interior of R. With the help of geometric properties of this problem we will be able to give algorithms to solve (ROL) for a large class of possible sets R.

The remainder of the paper is organized as follows: First we state geometrical properties of level curves and level sets. These results are used to derive a combinatorial description of the solution set in $\mathbb{R}^n$ for (ROL) and convex sets R. Section 4 shows how these results can be extended in the plane for more general sets R while maintaining the combinatorial character of the solution. The paper ends with some conclusions and an outview to further research.

2. The concept of cells and level sets

For any set $R \subseteq \mathbb{R}^n$ let $\mathrm{ext}(R)$ denote the extreme points of R and ∂R the boundary of R. Furthermore, let $\mathcal{K} = \{1, 2, \ldots, K\}$. Then $f(x)$ can be written as $f(x) = \max_{k \in \mathcal{K}}\{f_k(x)\}$, where for all $k \in \mathcal{K}$

$$f_k : \mathbb{R}^n \to \mathbb{R}$$

is an affine linear function and the f_k are pairwise different. To denote the solution sets we introduce

$$\begin{aligned} t^* &:= \min_{x\in\mathbb{R}^n} f(x) \\ t^*_R &:= \min_{x\in \mathrm{cl}(\mathbb{R}^n\setminus R)} f(x) \\ \mathcal{X}^* &:= \{x \in \mathbb{R}^n : f(x) = t^*\} \text{ and} \\ \mathcal{X}^*_R &:= \{x \in \mathbb{R}^n : x \notin \mathrm{i}nt(R) \text{ and } f(x) = t^*_R\}. \end{aligned}$$

By defining cells

$$C_i := \begin{cases} \{x\in\mathbb{R}^n : f_i(x) \geqslant f_k(x), \forall k \in \mathcal{K}\} & \text{if } \exists x \in \mathbb{R}^n : f_i(x) > f_k(x), \forall k\in\mathcal{K}, k \neq i \\ \emptyset & \text{else} \end{cases}$$

we get a subdivision of $\mathbb{R}^n$ into not more than K nonempty cells. The set of all cells is denoted $\mathcal{C}$. Some properties of $\mathcal{C}$ are summarized in the following lemma.

LEMMA 1. *For the cells C_i, $i \in \mathcal{K}$ the following hold:*

1. *For $i \in \mathcal{K}$ we have that either C_i is a polyhedral set with dimension n or $C_i = \emptyset$.*
2. $\mathrm{int}(C_i) = \{x \in \mathbb{R}^n : f_i(x) > f_k(x), \forall k \in \mathcal{K}, k \neq i\}$.
3. $\bigcup_{i\in\mathcal{K}} C_i = \mathbb{R}^n$
4. $\mathrm{int}(C_i) \cap \mathrm{int}(C_j) = \emptyset \ \ \forall i, j, \in \mathcal{K}$ *with* $i \neq j$.

Proof.

ad 1) From the definition of C_i it follows that C_i is either empty or a polyhedral set which is full-dimensional, because in that case $\mathrm{int}(C_i) \neq \emptyset$.

ad 2) Let $x \in \mathrm{int}(C_i)$ and suppose there exists $k \neq i$ such that $f_i(x) = f_k(x)$. As x is in the interior of C_i there exists a ball $U := U(x)$ around x such that $U(x) \subseteq \mathrm{int}(C_i)$. As $f_i \neq f_k$ the hyperplane $H := \{x : f_i(x) = f_k(x)\}$ separates U into two parts U^+_H, U^-_H such that for all $y \in U^+_H$ $f_i(y) > f_k(y)$ and $f_i(y) < f_k(y)$ for all $y \in U^-_H$. This is a contradiction since $U \subseteq C_i$.

ad 3) Take an $x \in \mathbb{R}^n$. We then have an index i such that $f_i = \max\{f_1, f_2, \ldots, f_K\}$. If i is unique, $x \in \mathrm{int}(C_i)$ and we are done. If i is not unique we define

$$\mathcal{I} := \{i \in \mathcal{K} : f_i(x) \geqslant f_k(x) \text{ for all } k \in \mathcal{K}\}$$

Then there exists a ball $U := U(x)$ around x such that

$$\forall y \in U : f_k(y) < f_i(y) \text{ for all } k \notin \mathcal{I}$$

Now define for all $k, l, k \neq l, k, l \in \mathcal{I}$: $g_{kl} := \{y \in U : f_k(y) = f_l(y)\}$ and G as the union of all g_{kl}. As the g_{kl} are hyperplanes, $U \setminus G \neq \emptyset$. Take $z \in U \setminus G$. Then there exists a unique $j \in \mathcal{I}$ such that

$$f_j(z) < f_k(z) \forall k \in \mathcal{K} \setminus \mathcal{I} \text{ and } f_j(z) < f_k(z) \forall k \in \mathcal{I}, k \neq j,$$

meaning that the cell $C_j \neq \emptyset$. As also $f_j(x) \geqslant f_k(x)$ for all $k \in \mathcal{K}$ we can conclude that $x \in C_j$.

ad 4) This statement follows directly from 2. □

As the lemma shows, we need only to investigate cells $C_i \neq \emptyset$ and we can neglect the empty ones. In the following we therefore will assume that all cells are non-empty. The concept of cells has frequently been used in optimization (see [19, 3, 9, 16] and references therein). From this literature the following result is well-known, but restated here for the specific context of this paper.

THEOREM 1. *$\mathcal{X}^*$ is either a whole cell or an r-dimensional facet of a cell,* $r \in \{0, 1, \ldots, n-1\}$.

Proof. We can solve (OL) by inspecting all cells. For any cell C_i the optimization problem is a linear program of which the feasible set is the polyhedral set C_i. Therefore the result follows from the well known properties of linear programming theory. □

The following corollary says, that, for finding one optimal solution it is enough to look at the extreme points of the cells.

COROLLARY. *There always exists a point $x \in \mathcal{X}^*$ with x is the extreme point of a cell C_i for an index $i \in \mathcal{K}$.*

We define $\mathcal{C}^\partial$ as the set of all facets of all cells $C \in \mathcal{C}$.

Additionally, we define construction hyperplanes

$$H_{ij} := \left\{x \ : \ f_i(x) = f_j(x)\right\}$$

for all $i, j \in \mathcal{K}$ with $i \neq j$. The set of all construction hyperplanes, denoted by $\mathcal{H}$, constitutes an arrangement of hyperplanes which define a set of cells $\mathcal{C}_{\mathcal{H}}$ similar to the ones introduced by Edelsbrunner [3]. Note that $\mathcal{C}_{\mathcal{H}}$ is a subpartition of $\mathcal{C}$, i.e. $\mathcal{P}(\mathcal{C}^\partial) \subseteq \mathcal{P}(\mathcal{H})$ (where $\mathcal{P}(\mathcal{C}^\partial)$ and $\mathcal{P}(\mathcal{H})$ denote the set of points $x \in \mathbb{R}^n$ belonging to a facet of a cell or a construction hyperplane, respectively).

In the 2-dimensional case we have 1-dimensional construction hyperplanes. In this case we will refer to them as ‘construction lines’.

EXAMPLE 1. We are given $f = \max\{f_1, \ldots, f_5\}$, where $f_1 = x_1 + x_2 - 20$, $f_2 = x_1 - x_2$, $f_3 = -x_1 + x_2$, $f_4 = -x_1 - x_2 + 20$ and $f_5 = \frac{1}{2}x_1 + 10$. In Figure 1 the two cell partitions $\mathcal{C}$ and $\mathcal{C}_{\mathcal{H}}$ are shown. We can easily compute $\mathcal{X}^*$ by checking the extreme points a_1, a_2, a_3 and a_4 of all cells $C \in \mathcal{C}$. By comparison of the objective values we get $\mathcal{X}^* = a_1$, given by the coordinates $(0, 10)$ with objective value 10.

One more definition is necessary for the paper. For $t \in \mathbb{R}$ define the level set $L_{\leqslant}(t)$ and the level curve $L_=(t)$ as

$$L_{\leqslant}(t) = \{x \in \mathbb{R}^n : f(x) \leqslant t\},$$
$$L_=(t) = \{x \in \mathbb{R}^n : f(x) = t\}.$$

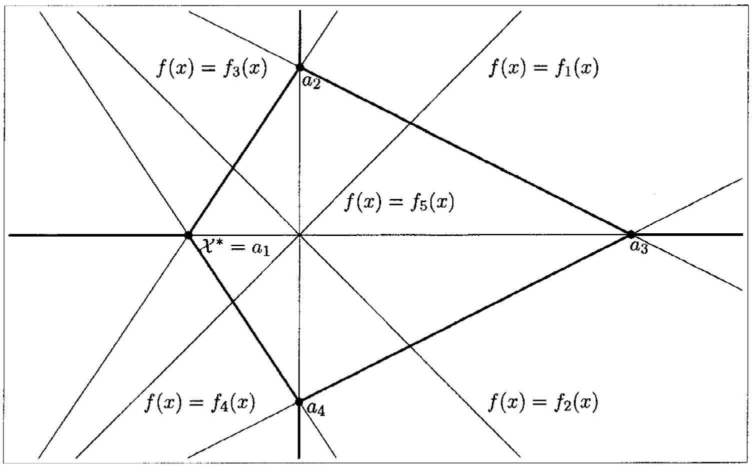

Figure 1. Illustration for Example 1. The bold part shows $\mathcal{C}^\partial$. The normal and the bold lines together constitute $\mathcal{H}$.

Note that for any convex function f, $L_\leqslant(t)$ is a closed, convex set, which is nonempty for all $t \geqslant t^*$.

LEMMA 2. *For $t > t^*$ we have that $\partial L_\leqslant(t) = L_=(t)$ and for $t \leqslant t^*$ $L_\leqslant(t) = L_=(t)$.*

Proof. For $t \leqslant t^*$ the result is obvious. For $t > t^*$ we know that any convex function f with minimal value t^* is strictly convex on the set $\{x \in \mathbb{R}^n : f(x) > t^*\}$ of all non-minimal solutions. That proves the lemma. □

Using level curves and level sets we can reformulate (OL) and (ROL).

THEOREM 2.

(a) t^* is the optimal objective value of (OL)
$\Leftrightarrow t^* = min\{t : L_=(t) \neq \emptyset\}$

(b) t_R^* is the optimal objective value of (ROL)
$\Leftrightarrow t_R^* = min\{t : L_=(t) \cap A \neq \emptyset\}$

(c) In (a) and (b) $L_=(t)$ can be replaced by $L_\leqslant(t)$

(d) x is an optimal solution of (ROL) with $f(x) = t_R^*$ if and only if there exists a $t \in \mathbb{R}$, such that

$$L_=(t) \cap \partial R \neq \emptyset \tag{1}$$

and

$$L_\leqslant(t) \subseteq R \tag{2}$$

The proof follows easily from the definition of level curves, level sets, and Lemma 2.

LEMMA 3. *The level curves $L_=(t)$ are piecewise linear on $\mathcal{C}$. More specific, $L_=(t) \cap C_i$ always has one of the following shapes:*

1. *empty*
2. *the whole cell C_i, in this case we have that $t = t^*$*
3. *an r-dimensional facet of C_i, $r \in \{0, 1, \dots, n-1\}$*
4. *the intersection between C_i and a hyperplane H with $\text{int}(C_i) \cap H \neq \emptyset$, then it follows that $t > t^*$*

Note that only in Case 2 and Case 4 we can say something about the optimality of t.

Proof.

$$\begin{aligned} L_=(t) \cap C_i &= \{x \in C_i : f(x) = t\} = \{x \in C_i : f_i(x) = t\} \\ &= C_i \cap \{x \in \mathbb{R}^n : f_i(x) = t\} \\ &=: C_i \cap H \end{aligned}$$

Case 1: $f_i(x) = const \;\; \forall x \in \mathbb{R}^n$. Then H can either be empty (yielding the first case) or $H = \mathbb{R}^n$ yielding that $L_=(t) \cap C_i = C_i$ and $f_i(x) = const = t$. In the latter case we also have that for all $x \in C_i$ $f(x) = f_i(x) = t$. Since f is a convex function, it only can be constant on a full-dimensional set if it is minimal on that set. Therefore that we can conclude $t = t^*$ and the whole cell C_i is optimal.

Case 2: $f_i(x)$ is not a constant function. Then H is a hyperplane in $\mathbb{R}^n$ and $L_=(t) \cap C_i = H \cap C_i$.

- If we have that $H \cap C_i \subseteq \partial C_i$ then H is a supporting hyperplane for C_i and, since C_i is a polyhedral set it follows that $H \cap C_i$ is a facet of C_i.
- If, otherwise, $L_=(t) \cap C_i \not\subseteq \partial C_i$, we can conclude that $\text{int}(C_i) \cap L_=(t) \neq \emptyset$. In this case let C_i^+ and C_i^- be the two parts of the cell C_i which are separated by the hyperplane H. That means, one of C_i^+,C_i^- is completely contained in $L_\leqslant(t)$. Let $C_i^+ \subseteq L_\leqslant(t)$ and suppose $t = t^*$. Consequently, $C_i^+ \subseteq L_\leqslant(t^*) = L_=(t^*)$ (see Lemma 2), which is a contradiction, because $C_i^+ \not\subseteq H \cap C_i = L_=(t) \cap C_i$. □

Note that as C_i is convex we have that

$$x, y \in L_=(t) \cap C_i \Longrightarrow \forall \lambda \in [0, 1] : \quad \lambda x + (1-\lambda) y \in L_=(t) \cap C_i$$

COROLLARY. *If $L_=(t) \cap C_i \neq \emptyset$ then $L_=(t) \cap \partial C_i \neq \emptyset$.*

Proof. As $L_=(t) \cap C_i \neq \emptyset$ there are three possibilities according to Lemma 3.

- If $L_=(t) \cap C_i = C_i$, then $\partial C_i \subseteq L_=(t)$ and, consequently, $L_=(t) \cap \partial C_i \neq \emptyset$.
- If $L = (t) \cap C_i$ is a facet of C_i then $L_=(t) \cap C_i \subseteq \partial C_i$.
- If there exists a hyperplane H with $C_i \cap L_=(t) = C_i \cap H$ and $\text{int}(C_i) \cap L_=(t) \neq \emptyset$ then suppose $L_=(t) \cap \partial C_i = \emptyset$. Then

$$\begin{aligned} & L_=(t) \subseteq \text{int}(C_i) \\ \Longrightarrow & L_\leqslant(t) \subseteq \text{int}(C_i) \\ \Longrightarrow & \mathcal{X}^* \subseteq \text{int}(C_i) \end{aligned}$$

which is a contradiction to Theorem 1. □

The following theorem is a consequence of the convexity of f (see also [11] and [8]).

THEOREM 3. *If* $\mathcal{X}^* \subseteq \text{int(R)}$ *then we have* $\mathcal{X}^*_R \subseteq \partial R$

Proof. Let $x \in \mathcal{X}^*$ and assume that there exists $y \in \mathbb{R}^n$ such that $y \in \mathcal{X}^*_R \setminus \partial R$. Then $y \notin R$, yielding that $f(x) < f(y)$ and therefore we know for all $\lambda \in (0, 1]$ that

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\leqslant \lambda f(x) + (1-\lambda) f(y) \\ &< \lambda f(y) + (1-\lambda) f(y) \\ &= f(y). \end{aligned}$$

This also holds if we choose λ such that $\lambda x + (1-\lambda)y \in \partial R$, which proves the theorem. □

An equivalent formulation and some more properties of (OL) and (ROL) can be found in [20], Section 1.5.

3. Solution for convex forbidden sets in $\mathbb{R}^n$

In this section we look at forbidden sets R which are convex. As we will show in the following theorem, for these sets there always exists an optimal solution for the restricted problem, which is in the intersection of the boundary of R with the the boundary of a cell $C \in \mathcal{C}$:

THEOREM 4. *Let R be a convex set, $R \subseteq \mathbb{R}^n$ and $\mathcal{X}^* \subseteq \text{int}(R)$. Then there exists $x^*_R \in \mathcal{X}^*_R$ and $c \in \mathcal{C}^\partial$ with*

$$x^*_R \in (c \cap \partial R) \ .$$

Furthermore only intersections with $\dim(c \cap \partial R) < n - 1$ *have to be investigated.*

Proof. Let $x \in \partial R \cap C_i \cap \mathcal{X}^*_R$ such that $x \notin \mathcal{C}^\partial$. Let $t = f(x)$. Since $\mathcal{X}^* \subseteq \text{int}(R)$ we know that $t > t^*$, and consequently (see Lemma 3)

$$H_i := \{y \in \mathbb{R}^n : f_i(y) = t\}$$

is a hyperplane with $L_=(t) \cap C_i = H_i \cap C_i$. Now we look at the following two cases.

Case 1: $H_i \cap C_i \subseteq R$. Note that since $x \notin \mathcal{C}^\partial$ we know that $x \in \text{int}(H_i \cap C_i)$. Therefore we can conclude that $H_i \cap C_i \subseteq \partial R$, as R is convex, H_i a hyperplane, and $x \in \partial R \cap H_i$. According to the corollary to Lemma 3 there exists $z \in L_=(t) \cap \partial C_i \subseteq \partial R$ such that $z \in \mathcal{X}^*_R \cap (\partial R \cap \mathcal{C}^\partial)$, $\partial R \cap \mathcal{C}^\partial \neq \emptyset$.

Case 2: $H_i \cap C_i \not\subseteq R$. Then there exists $z \in H_i \cap C_i$ with $z \notin R$ and $f(z) = f(x)$. That means $t = f(x)$ cannot be optimal for the restricted problem since Theorem 3 tells us that $\mathcal{X}_R^* \subseteq \partial R$.

Now suppose we have $\dim(c \cap \partial R) = n - 1$ for a $c \in \mathcal{C}^\partial$. This implies that c coincides locally with ∂R and among c the objective function is linear. Assume that there is no $c' \in \mathcal{C}^\partial$, $c \neq c'$ with $(c \cap \partial R) \cap c' \neq \emptyset$. Then every level curve $L_=$ touching $c \cap \partial R$ has to be linear throughout $c \cap \partial R$ and therefore $L_= \not\subseteq R$, contradicting the optimality condition. It follows that for an $x \in (c \cap \partial R)$ to be optimal there must exist a $c' \neq c$, $c' \in \mathcal{C}^\partial$ with $(c \cap \partial R) \cap c' \neq \emptyset$. Since $\dim(c \cap c') < n - 1$ it follows that $\dim(c' \cap \partial R) < n - 1$ and the result follows by replacing c by c'. □

THEOREM 5. *Let $R \subseteq \mathbb{R}^n$ be a convex polyhedron, and $\mathcal{X}^* \subseteq \mathrm{int}(R)$. The set of facets of R is denoted by $\mathcal{F}$. Then there always exists an optimal solution $x_R^* \in \mathcal{X}_R^*$ in the finite set of points*

$$\begin{aligned} Cand := \{A_1 \cap A_2 \cap \ldots \cap A_n \ : \ & A_i \in (\mathcal{F} \cup \mathcal{H}),\ A_i \neq A_j \textit{ for } i \neq j \\ & \textit{dim}(A_1 \cap A_2 \cap \ldots \cap A_n) = 0 \textit{ and} \\ & \exists i \neq j \textit{ with } A_i \in \mathcal{F} \textit{ and} A_j \in \mathcal{H}\}\ . \end{aligned}$$

Proof. From Theorem 4 we know that for an optimal x^* we have $x^* \in (\partial R \cap H)$ for some $H \in \mathcal{H}$. Choose a facet $F \in \mathcal{F}$ such that $x^* \in (F \cap H)$. Furthermore, choose an index i such that also $x^* \in \partial C_i$ holds. Now we have $x^* \in (F \cap H \cap C_i)$. Therefore $f = f_{C_i} \in F \cap H \cap C_i$, which means f is linear in $F \cap H \cap C_i$. Now we can conclude from the theory of linear programming that there always exists an optimal vertex v of the polyhedron $F \cap H \cap C_i$ with $f(v) \leqslant f(x^*)$. But for v we know that $v \in H \cap \partial R$ and additionally $\dim(v) = 0$. □

The next theorem gives a characterization of the optimal solution for all convex restricted sets. Note that in $\mathbb{R}^2$ this result is equivalent to Theorem 4.

THEOREM 6. *Let $R \subseteq \mathbb{R}^n$ be convex , and $\mathcal{X}^* \subseteq \mathrm{int}(R)$. Then there always exists an optimal solution $x_R^* \in \mathcal{X}_R^*$ such that x_R^* is a zero-dimensional intersection between the boundary ∂R and a sufficient number of construction hyperplanes $H \in \mathcal{H}$.*

Proof. From Theorem 4 we know that there always exists an optimal solution $x_R^* \in \mathcal{X}_R^*$ such that $x_R^* \in \partial R \cap H^1$ for

$$H^1 = \{x \in \mathbb{R}^n : f_a(x) = f_b(x)\} \in \mathcal{H}.$$

If $\partial R \cap H^1$ only consists of one single point, we are done. If not, we consider the following problem (ROL1) on the $n - 1$ dimensional space H^1:

$$\text{(ROL1)} \quad \min f(x)$$
$$s.t.\ \ x \in H^1 \setminus (R \cap H^1)$$

Note that the optimal solutions of (ROL1) are optimal for (ROL). As the restriction of f to the linear subspace H^1 remains piecewise linear and convex, $R \cap H^1$ is a convex set, and

$$\begin{aligned}\mathcal{H}^1 &= \{H \cap H^1 : H \in \mathcal{H} \setminus H^1\}\\ &= \{\{x \in H^1 : f_i(x) = f_j(x)\} : i \neq j, i, j \in \{1, 2, \dots, K\} \setminus \{a, b\}\}\end{aligned}$$

are the new construction hyperplanes we can apply Theorem 4 again. We conclude that there always exists an optimal solution x_R^{*1} to (ROL1) (and therefore to (ROL)) such that $x_R^{*1} \in \partial(R \cap H^1) \cap H$ for a $H \in \mathcal{H}^1$. Let $H = H^2 \cap H^1$ with $H^2 \in \mathcal{H}$ according to the definition of $\mathcal{H}^1$. Then we know that

$$x_R^{*1} \in \partial R \cap H^1 \cap H^2 \text{ with } H^1, H^2 \in \mathcal{H}.$$

We repeat this argument until we have a number z of hyperplanes $H^1, H^2, \dots, H^z \in \mathcal{H}$ such that the set

$$\partial R \cap H^1 \cap H^2 \dots \cap H^z$$

only consists of one single point, which then is an optimal solution to (ROL). □

The following corollary shows how Theorems 4 and 6 can be used to derive an efficient algorithm for solving (ROL) in the plane. We use the fact that $\mathcal{P}(\mathcal{C}^\partial) \subseteq \mathcal{P}(\mathcal{H})$ since $\mathcal{H}$ is easier to compute.

COROLLARY. *Let $n = 2$ and $R \subseteq \mathbb{R}^2$ be a convex set such that $\mathcal{X}^* \subseteq \text{int}(R)$. Then there always exists an optimal solution $x_R^* \in \mathcal{X}_R^*$ in the finite set of points*

$$Cand = \{H \cap \partial R : H \in \mathcal{H} \textit{ and } \dim(H \cap \partial R) = 0\}$$

In the algorithm we first compute the set $Cand$ and then look for the best candidate.

ALGORITHM for solving (ROL) in the plane.

Input : $f_1, f_2, \dots, f_K$, convex set R

Output : $\mathcal{X}_R^*$

0. $Cand = \emptyset$.

1. Compute $\mathcal{X}^*$. If $\mathcal{X}^* \cap A \neq \emptyset$ then Output: $\mathcal{X}_R^* := \mathcal{X}^* \cap A$.

2. Compute $\mathcal{H}$.

3. For all $H \in \mathcal{H}$ compute $H \cap \partial R$. If $\dim(H \cap \partial R) = 0$ then

$$Cand = Cand \cup \{H \cap \partial R\}$$

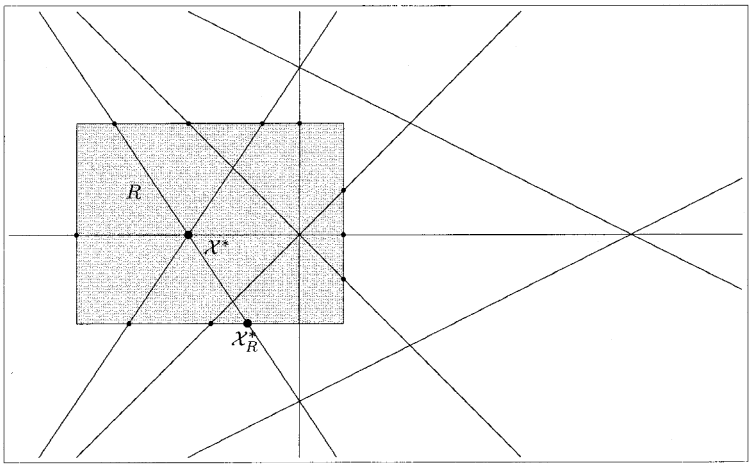

Figure 2. Illustration for Example 2. The bold points show the set of candidates $\mathcal{H} \cap \partial R$.

4. Determine $t_R = \min_{P \in Cand}\{f(P)\}$.

5. Output: $\mathcal{X}_R^* = L_=(t_R^*) \cap \partial R$.

The complexity of the above algorithm is dominated by the complexity of Step 3 which is $Int\ O(K^2)$ (where Int is the complexity of computing an intersection point between a line and the boundary of the forbidden set R) and the complexity of Step 4 being $O(K^3)$. Overall we have a complexity of $O(K^3) + Int\ O(K^2)$.

EXAMPLE 2. We use the same objective function f as in Example 1. In addition we are given a convex set $R = [-10, 14] \times [2, 20]$. By checking the set of candidates $Cand$ we get $\mathcal{X}_R^* = (\frac{16}{3}, 2)$ with objective value $\frac{38}{3}$ (see Figure 2). In Figure 3 the reduced candidate set based on Theorem 4 is shown.

When applying Theorems 4 and 5 for higher dimensions to derive efficient algorithms a pure enumeration of the set of Candidates is not appropriate. Therefore combined enumeration and search procedures are suggested. Instead of solving

$$\begin{aligned} P_F &: \min z \\ s.t. &\quad f_i(x) \leqslant z \quad i = 1, 2, \ldots, K \\ &\quad x \in F \end{aligned}$$

for all facets $F \in \mathcal{F}$ it is also possible to restrict the search procedure to linear subspaces of lower dimensions and solve for all $H = H_{ij} = \{x : f_i(x) = f_j(x)\}$ and all facets $F \in \mathcal{F}$

$$\begin{aligned} P_{HF} &: \min f_i(x) = c_i x \\ s.t. &\quad x \in H \cap F \end{aligned}$$

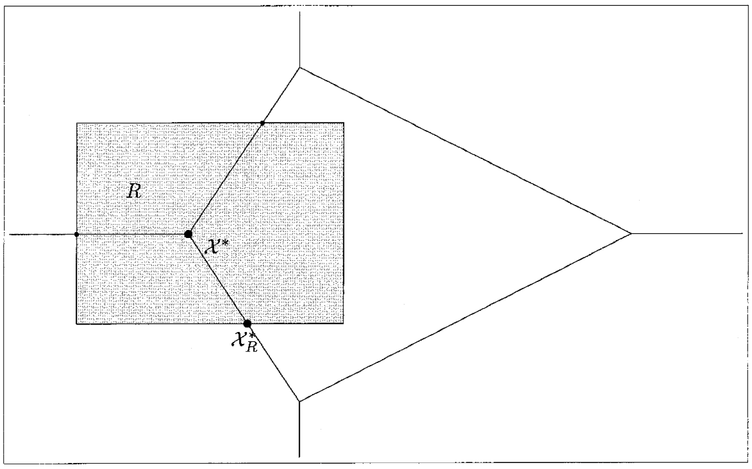

Figure 3. Illustration for Example 2. The bold points show the set of candidates $\partial\mathcal{C} \cap \partial R$.

Which of those alternative solution approaches is more appropriate is dependent on the input data.

4. Extensions in the plane

In this section we consider more realistic forbidden sets. At first we examine what happens, if the forbidden set is not connected, and then extend the results to so-called bumpy sets.

The algorithm of the previous section can easily be modified to accommodate the case where R is the union of pairwise disjoint, convex sets $R_1, R_2, \ldots, R_L$ (i.e. $R_i \cap R_j = \emptyset$ for $i \neq j, i, j \leqslant L$).

In this situation we first solve the unrestricted problem (OL). Then the following result is an immediate consequence of Theorem 3.

THEOREM 7. *Let $R = R_1 \cup R_2 \cup \ldots \cup R_L$ where $R_i \cap R_j = \emptyset$ for $i \neq j, i, j \leqslant L$ and R_i are convex sets for $i = 1, \ldots, L$. Then there exists an optimal solution x_R^* such that either $x_R^* \notin int(R)$ or there exists some l with $\mathcal{X}_R^* \subseteq \partial R_l$.*

Proof. If $\mathcal{X}^* \subseteq int(R)$, then the convexity of $\mathcal{X}^*$ and the assumptions on the sets $R_1, R_2, \ldots, R_L$ imply that there exists some l such that $\mathcal{X}^* \subseteq R_l$. Hence we can replace R by R_l and use Theorem 3 to conclude that $\mathcal{X}_R^* \subseteq \partial R_l$. □

Our results can be extended to other cases of non-convex restricting sets R. If we review the proof of Theorem 4 it becomes apparent that we can apply a combinatorial algorithm as in the case of convex sets whenever we have only a finite set of points on ∂R which can lie in the (relative) interior of linear pieces of a level curve. Therefore we introduce the concept of bumpy sets, which was first mentioned in [9] in the area of location theory.

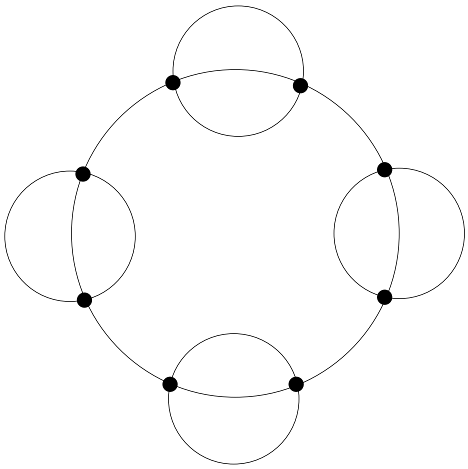

Figure 4. An example for a bumpy set. The dots mark a set of roots.

DEFINITION 1. A *bumpy set* $R \subseteq \mathbb{R}^2$ with its set of roots *Roots(R)* is any set which can be constructed by a finite application of the following rules:

1. Any convex set R is a bumpy set. In this case $Roots(R) = \emptyset$.
2. If R_0 is a bumpy set then $R = R_0 \cup R_1$ is a bumpy set, if
   - $conv(R_1) \subseteq R$ and
   - $(\partial R_0) \cap R_1 =: s$ is a one-dimensional, connected, proper curve segment (i.e. there exists a bijective mapping $\phi : [0, 1] \to S$)

   In this case R_0 is called the base of R and R_1 the bump. The two disjoint endpoints r_1, r_2 of s are the new roots of R, i.e.

$$\mathit{Roots}(R) = \mathit{Roots}(R_0) \cup \{r_1, r_2\}.$$

Note that depending on the construction of the bumpy set, the same bumpy set can have a different set of roots, i.e. $Roots(R)$ is not uniquely defined. For the following algorithm, however, any set of roots leads to an optimal solution. For the running time, the smallest set of roots is preferable.

In Figure 4 and Figure 5 some examples of bumpy sets are shown, and the following lemmas describe some classes of bumpy sets. Some more applications of bumpy sets will be shown at the end of this section.

LEMMA 4. *Let R be a strictly convex set and R' a translate of R, such that $int(R \cap R') \neq \emptyset$. Then $R = R \cup R'$ is a bumpy set.*

Proof. Choose R as base and R' as bump.

- R is convex and therefore a bumpy set,
- R' is convex and therefore $conv(R') \subseteq R \cup R'$ and
- $\partial R \cap R'$ is a one-dimensional, connected, proper line segment as $\partial R \cap \partial R'$ consists of only two points (see, e.g. [12]). □

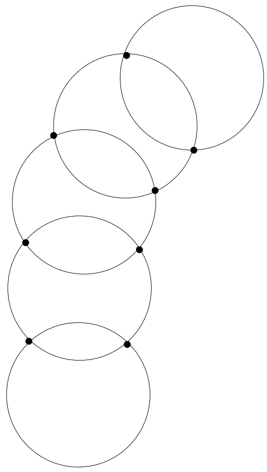

Figure 5. Another example for a bumpy set. The dots mark a set of roots.

LEMMA 5. *Let R_1, R_2 be convex sets with $int(R_0 \cap R_1) \neq \emptyset$. Then $R = R_0 \cup R_1$ is a bumpy set.*

Proof. If R is convex it trivially is a bumpy set. If not we define the convex set R_0 as the base of the bump. Let $B_1, B_2, \ldots B_L$ be the connected components of $R_1 \setminus R_0$. As R is not convex, we know that $R_0 \not\subseteq R_1$ and therefore $(\partial R_0) \cap R_1$ consists of L connected, one-dimensional, proper curve segments. That means we can iteratively add all bumps $B_1, B_2, \ldots, B_L$ to R_0, as we know that $conv(B_l) \subseteq R_0 \cup B_l$ for all $l = 1, \ldots, L$. □

As mentioned, we now will extend the algorithm of the previous section to the case, where R is a bumpy set. In the following we therefore prove that to find an optimal solution to (ROL), it again is enough to evaluate a finite candidate set. In order to do this one more result about the structure of bumpy sets is needed.

LEMMA 6. *Let R be a bumpy set and s be a line segment. If $s \subseteq R$ and $int(s)$ touches ∂R at a unique point x then $x \in Roots(R)$.*

Proof. Induction over the number of bumps.

For a convex set R and a line segment $s \subseteq R$ we know that the interior of s cannot touch ∂R from inside at a unique point.

Now take any bumpy set R with base R_0 and bump R_1. Let s be a line segment with its interior touching ∂R from inside at a unique point, i.e., $s \subseteq R$ and $int(s) \cap \partial R = \{x\}$. Now we distinguish three cases:

- If $x \notin \partial R_1$, we know that $x \in Roots(R_0) \subseteq Roots(R)$ due to the induction hypothesis.
- If $x \in \partial R_1 \cap R_0$, then by definition $x \in \partial R$ is one of the (new) roots of R.
- If $x \in \partial R_1 \setminus R_0$ then $x \in \partial conv(R_1)$ as $conv(R_1) \subseteq R$ and therefore we again have a linear piece touching a convex set from inside at a unique point, which is a contradiction. □

The following theorem shows that in order to solve (ROL) with R is a bumpy set it suffices to examine the candidate set of the last section, i.e. the zero-dimensional intersections between the boundary of R and the construction lines $H \in \mathcal{H}$ and additionally, the roots of the bumpy set R.

THEOREM 8. *Let $R \subseteq \mathbb{R}^2$ be a bumpy set such that $\mathcal{X}^* \subseteq \mathrm{int}(R)$. Then there always exists an optimal solution $x_R^* \in \mathcal{X}_R^*$ in the finite set of points*

$$Cand_{bumpy} = Cand \cup Roots(R)$$

Proof. With Theorem 2 we know that t_R^* is the optimal value of (ROL) if and only if $L_\leqslant(t_R^*) \subseteq R$ and $L_=(t_R^*) \cap \partial R \neq \emptyset$. We also know that $L_=(t_R^*) \cap \partial R$ is the set of optimal solutions. Now take any $x_R^* \in L_=(t_R^*) \cap \partial R$.

Case 1: There exists an open set $U \in \mathbb{R}^2$ such that we have a unique point $x_R^* \in U$, where $L_=(t_R^*)$ touches ∂R, i.e.

$$\{x_R^*\} = L_=(t_R^*) \cap \partial R \cap U.$$

As the level curve $L_=(t_R^*)$ consists of linear pieces (see Lemma 3) we either have

- that x_R^* is the endpoint of such a linear piece, in this case $x_R^* \in H \cap \partial R$ for some $H \in \mathcal{H}$ (see Theorem 4) or
- x_R^* is in the interior of such a linear piece, then, according to Lemma 6, $x_R^* \in Roots(R)$.

Case 2: x_R^* is part of a linear piece $s \subseteq L_=(t_R^*) \cap \partial R$. Then, as $L_\leqslant(t_R^*) \subseteq R$ we know that either there exists a root $x \in s$ or the endpoints x_1, x_2 of s are in ∂R (see [20] for more details). In the first case, we note that $f(x) = t_R^*$ and $x \in Cand_{bumpy}$ is the candidate point we are looking for. In the latter case we clearly have $f(x_1) = f(x_2) = t_R^*$. At both endpoints, on the other hand, other construction lines $H_1, H_2 \in \mathcal{H}$ intersect (as the level curves are piecewise linear on the cells, see Lemma 3), such that we have $x_1, x_2 \in Cand_{bumpy}$. □

ALGORITHM for solving (ROL) in the plane with R is a bumpy set.

Input : $f_1, f_2, \dots, f_K$, bumpy set R

Output : $\mathcal{X}_R^*$

0. $Cand = \emptyset$.

1. Compute $\mathcal{X}^*$. If $\mathcal{X}^* \cap A \neq \emptyset$ then Output: $\mathcal{X}_R^* := \mathcal{X}^* \cap A$.

2. Compute $\mathcal{H}$.

3a. For all $H \in \mathcal{H}$ compute $H \cap \partial R$. If $\dim(H \cap \partial R) = 0$ then

$$Cand = Cand \cup \{H \cap \partial R\}$$

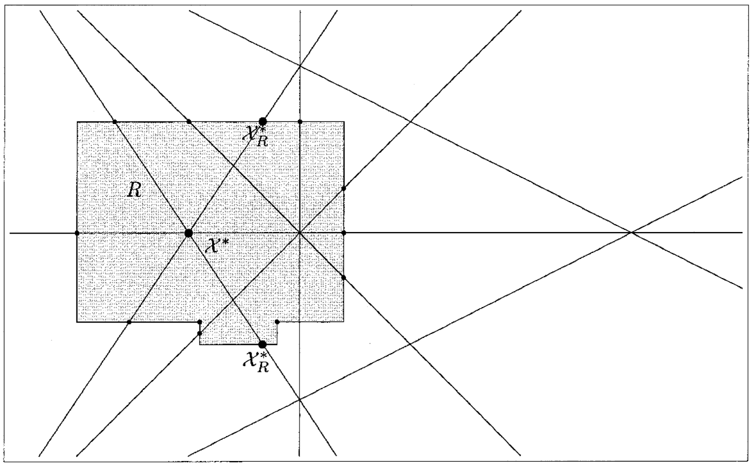

Figure 6. Illustration for Example 3. The bold points show the set of candidates $\mathcal{H} \cap \partial R$ plus the roots of the bump.

3b. $Cand = Cand \cup Roots(R)$

4. Determine $t_R = \min_{P \in Cand}\{f(P)\}$.

5. Output: $\mathcal{X}_R^* = L_=(t_R^*) \cap \partial R$.

EXAMPLE 3. We use the same objective function f and forbidden region R as in Example 2. In addition we are given a convex set $R_1 = [1, 8] \times [0, 2]$ as a bump. By checking the extended set of candidates $Cand$ we get the new candidates $(1, 2)$, $(8, 2)$, $(1, 1)$, and $(\frac{20}{3}, 0)$ and loose the two former candidates $(\frac{16}{3}, 2)$ and $(2, 2)$. By computing the objective function for the candidates we get $\mathcal{X}_R^* = \{(\frac{20}{3}, 0), (\frac{20}{3}, 20)\}$ with objective value $\frac{40}{3}$ (see Figure 6).

Finally we give two examples for the application of bumpy sets. The first application shows that we can solve the restricted problem (ROL) for any simple polygon. These simple polygons can then be used for approximating more complex sets R (see [1, 18 and 13] for more details).

LEMMA 7. *Any simple polygon is a bumpy set.*

Proof. Induction over the number of vertices n of the polygon. For $n = 3$ we have a convex triangle which trivially is a bumpy set. Now take any polygon R with more than 3 vertices. Consider a triangulation of R into $n - 2$ triangles (such a triangulation always exists, see e.g. [12, 17]) and take any triangle R_1 of that triangulation such that R_1 has two edges on the boundary of R. Define R_1 as bump and $R_0 := R \setminus R_1$ as base. Then the triangulation of R_0 consists of $n - 3$ triangles, such that the number of vertices of R_0 is $n - 1$. Therefore R_0 is a bumpy set, R_1 is convex and $s = \partial R \cap R_1$ is a one-dimensional, connected, proper line segment, since it is the third edge of the triangle R_1. $\square$

If R is a polygon then the set of roots in the above construction is contained in the set of vertices of the polygon, i.e.

$$Roots(R) \subseteq \{v : v \text{ vertex of } R\},$$

such that the set of candidates in Step 3b of the algorithm would be

$$Cand_{polygon} = Cand \cup \{v : v \text{ vertex of } R\}.$$

This can be sharpened, as the following lemma shows.

LEMMA 8. *Let R be a polygon and let $\mathcal{X}^* \subseteq \text{int}(R)$. Then there always exists an optimal solution $x^*_R \in \mathcal{X}^*_R$ in the finite set of points*

$$Cand_{polygon} = Cand \cup \{\textit{reflexive vertices}\}$$

Proof. This lemma can be shown by a special bumpy set construction using only reflexive vertices as roots. Another possibility is to prove the result along the lines of Theorem 8. □

Another application is the following. Suppose we want to have m solutions $x^{*1}, x^{*2}, \ldots, x^{*m}$ to the problem (OL), but have the restriction that these solutions are not too similar, i.e.

$$d(x^{*k}, x^{*l}) > r \text{ for all } k, l \in \{1, 2, \ldots, m\}, k \neq l$$

for a given number $r \in \mathbb{R}$ and a convex distance measure d. This concept has been used by Brimberg and Wesolowsky [2] in the context of location theory. Then we can proceed as follows: First we find the best solution x^{*1} to our problem. Then we forbid all solutions which are too similar to x^{*1}, i.e. $R^1 = \{x : d(x, x^{*1}) \leqslant r\}$. Solving that problem we find a solution $x^{*2} \in \partial R^1$. Iterating that procedure, to find x^{*k} we solve problem (ROL) with

$$\begin{aligned} R^{k-1} &= \{x : d(x, x^{*l}) \leqslant r \text{ for all } l \leqslant k-1\} \\ &= R^{k-2} \cup \{x : d(x, x^{*k-1}) \leqslant r\} \end{aligned}$$

Then $(x^{*1}, \ldots, x^{*m})$ is a lexicographic minimal solution to the above problem.

This problem can be solved with our algorithm, if all the sets R^k, $k = 1, \ldots, m-1$ are bumpy sets. If $m \leqslant 3$ Lemma 4 shows that this can be done for any strictly convex distance function d in $\mathbb{R}^2$. If we also want to have distance functions with linear pieces, e.g. l_1 or l_∞ distances or any kind of gauges we can use Lemma 5 for $m \leqslant 3$. For $m > 3$ it can be shown, that the sets R^k are bumpy sets, if they do not have any wholes. Even if they have wholes, Lemma 6 remains true, such that the algorithm can be adapted to solve the problem.

5. Conclusions

In this paper we have shown how global optimization problems can be solved using geometrical methods and combinatorial arguments. This discretization approach leads to very good results, especially if the original assumptions are weakened, like in the case of bumpy sets. Moreover, this approach yields numerically stable and efficient procedures and allows us to easily compute the whole set of optimal solutions. This has already been successfully done in location theory (see [15, 9] and references therein). This approach is not only useful for general piecewise linear problems, but furthermore, it can be extended to arbitrary convex objective functions using approximation methods, like the ones described in [1]. Therefore our future work will include the adaption of such approximation methods to (ROL). Also computational tests and bounding techniques are under consideration.

References

1. Burkard, R.E., Hamacher, H.W. and Günter Rote (1991). Sandwich approximation of univariate convex functions with an application to separable convex programming, *Naval Research Logistics Quarterly* 38: 911–924.
2. Brimberg, J. and Wesolowsky, G.O. (1996). The rectilinear distance minisum problem with minimum distance constraints, *Location Science* 3: 203–215.
3. Edelsbrunner, H. (1987). *Algorithms in Combinatorial Geometry*. Springer Verlag, Berlin/New York.
4. Francis, Richard L. Jr., McGinnis, Leon F. and White, John A. (1992). *Facility layout and location: An analytical approach*, 2nd edition. Prentice-Hall, New York.
5. Fourer, R. (1985). A simplex algorithm for piecewise-linear programming, I: derivation and proof, *Mathematical Programming* 33: 204–233.
6. Fourer, R. (1988). A simplex algorithm for piecewise-linear programming, II: finiteness, feasibility and degeneracy, *Mathematical Programming* 41: 281–315.
7. Fourer, R. (1992). A simplex algorithm for piecewise-linear programming, III: computational analysis and applications, *Mathematical Programming* 53: 213–235.
8. Hamacher, H.W. and Nickel, S. (1994). Combinatorial algorithms for some 1-facility median problems in the plane, *European Journal of Operational Research* 79: 340–351.
9. Hamacher, H.W. and Nickel, S. (1995). Restricted planar location problems and applications, *Naval Research Logistics* 42: 967–992.
10. Horst, R., Pardalos, P.M. and Thoai, N.V. (1995). *Introduction to Global Optimization*. Kluwer Academic Publishers, Dordrecht/Boston/London.
11. Horst, R. and Tuy, H. (1996). *Global Optimization*, 3rd edition. Springer Verlag, Berlin/New York.
12. Klein, Rolf (1997). *Algorithmische Geometrie*. Addison-Wesley.
13. Käfer, B. and Nickel, S. (1996). Error bounds for the approximative solution of restricted planar location problems. Technical report, University of Kaiserslautern, Department of Mathematics. Report in Wirtschaftsmathematik No. 17, submitted to *Computing*.
14. Love, Robert F., Morris, James G. and Wesolowsky, George O. *Facilities Location: Models & Methods*. North-Holland, Amsterdam/New York.
15. Nickel, S. and Hamacher, H.W. (1992). RLP: A program package for solving restricted 1-facility location problems in a user friendly environment, *European Journal of Operational Research* 62: 116–117.
16. Nickel, S. (1995). *Discretization of Planar Location Problems*. Shaker Verlag, Aachen.

17. O'Rourke, Joseph (1993). *Computational Geometry in C*. Cambridge University Press.
18. Rote, G. (1992). The convergence rate of the sandwich algorithm for approximating convex functions, *Computing* 48: 337–361.
19. Rourke, C.P. and Sanderson, B.J. (1972). *Introduction to piecewise-linear topology*. Springer Verlag, Berlin/New York.
20. Schöbel, A. (1999). *Locating Lines and Hyperplanes – Theory and Algorithms*. Applied Optimization. Kluwer, Academic Publishers, Dordrecht/Boston/London.